Static Program Analysis using Abstract Interpretation

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Introduction

Static Program Analysis

Static program analysis consists of automatically discovering properties of a program that hold for all possible execution paths of the program.

Static program analysis is **not**

- Testing: manually checking a property for some execution paths
- Model checking: automatically checking a property for all execution paths

Program Analysis for what?

- Optimizing compilers
- Program understanding
- Semantic preprocessing:
  - Model checking
  - Automated test generation
- **Program verification**

Program Verification

- Check that every operation of a program will never cause an error (division by zero, buffer overrun, deadlock, etc.)
- Example:

```
                  int a[1000];
                  for (i = 0; i < 1000; i++) {
safe operation  ──→ a[i] = … ;   // 0 <= i <= 999
                  }
buffer overrun ──→ a[i] = … ;     // i = 1000;
```

Incompleteness of Program Analysis

- Discovering a sufficient set of properties for checking every operation of a program is an undecidable problem!
- **False positives:** operations that are safe in reality but which cannot be decided safe or unsafe from the properties inferred by static analysis.

Precision versus Efficiency

Precision: number of program operations that can be decided safe or unsafe by an analyzer.

- Precision and computational complexity are strongly related
- Tradeoff precision/efficiency: limit in the average precision and scalability of a given analyzer
- Greater precision and scalability is achieved through **specialization**

Specialization

- Tailoring the program analyzer algorithms for a specific class of programs (flight control commands, digital signal processing, etc.)
- Precision and scalability is guaranteed for this class of programs only
- Requires a lot of try-and-test to fine-tune the algorithms
- **Need for an open architecture**

Soundness

- What guarantees the soundness of the analyzer results?
- In dataflow analysis and type inference the soundness proof of the resolution algorithm is independent from the analysis specification
- An independent soundness proof precludes the use of test-and-try techniques
- **Need for analyzers correct by construction**

Abstract Interpretation

- A general methodology for designing static program analyzers that are:
  - Correct by construction
  - Generic
  - Easy to fine-tune
- Scalability is difficult to achieve but the payoff is worth the effort!

Approximation

The core idea of Abstract Interpretation is the formalization of the notion of **approximation**

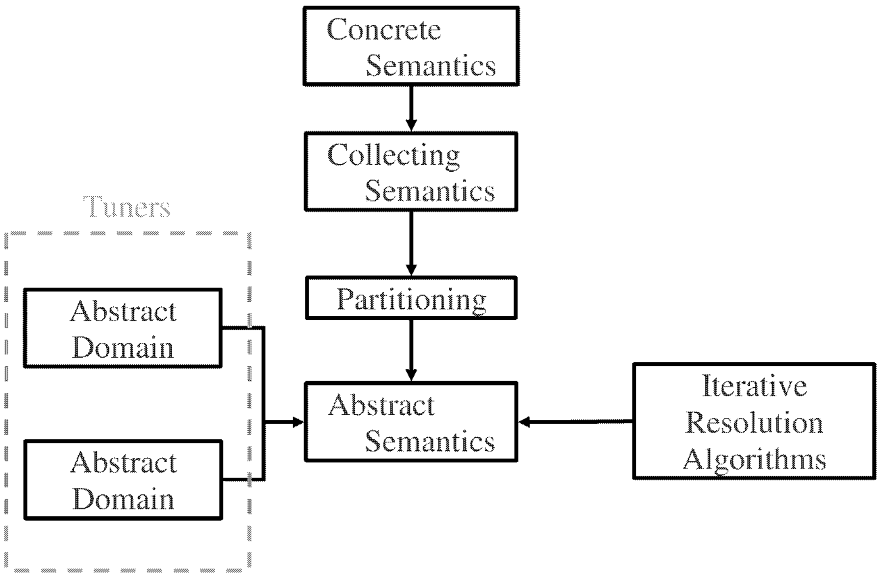
- An approximation of memory configurations is first defined
- Then the approximation of all atomic operations
- The approximation is automatically lifted to the whole program structure
- The approximation is generally a scheme that depends on some other parameter approximations

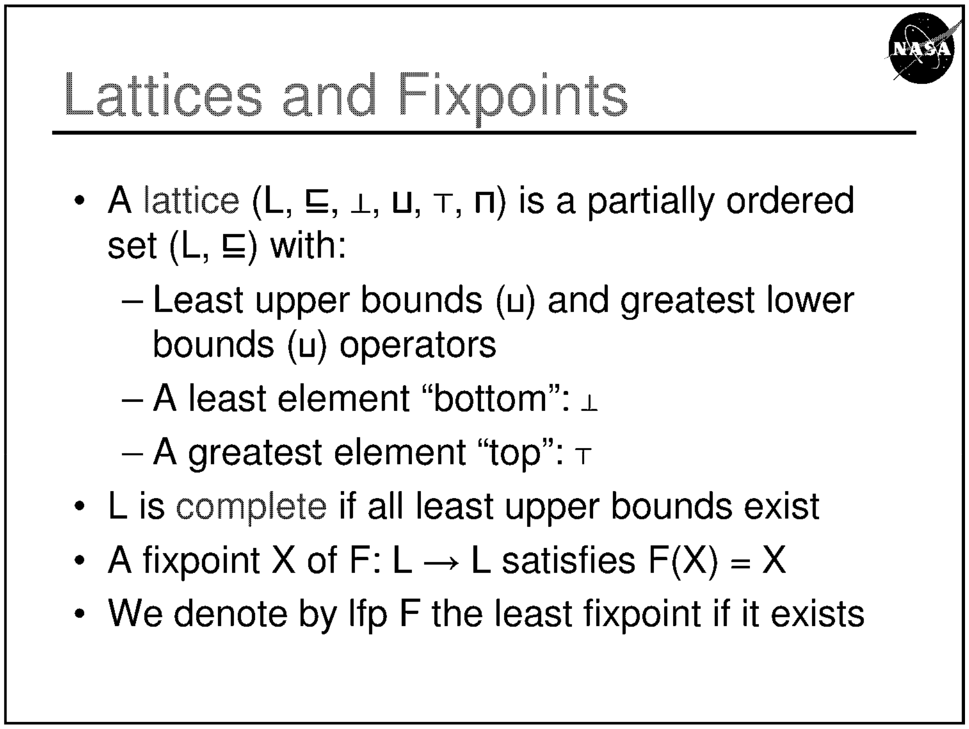
Overview of Abstract Interpretation

- Start with a formal specification of the program semantics (the concrete semantics)
- Construct abstract semantic equations w.r.t. a parametric approximation scheme
- Use general algorithms to solve the abstract semantic equations
- Try-and-test various instantiations of the approximation scheme in order to find the best fit

The Methodology of Abstract Interpretation

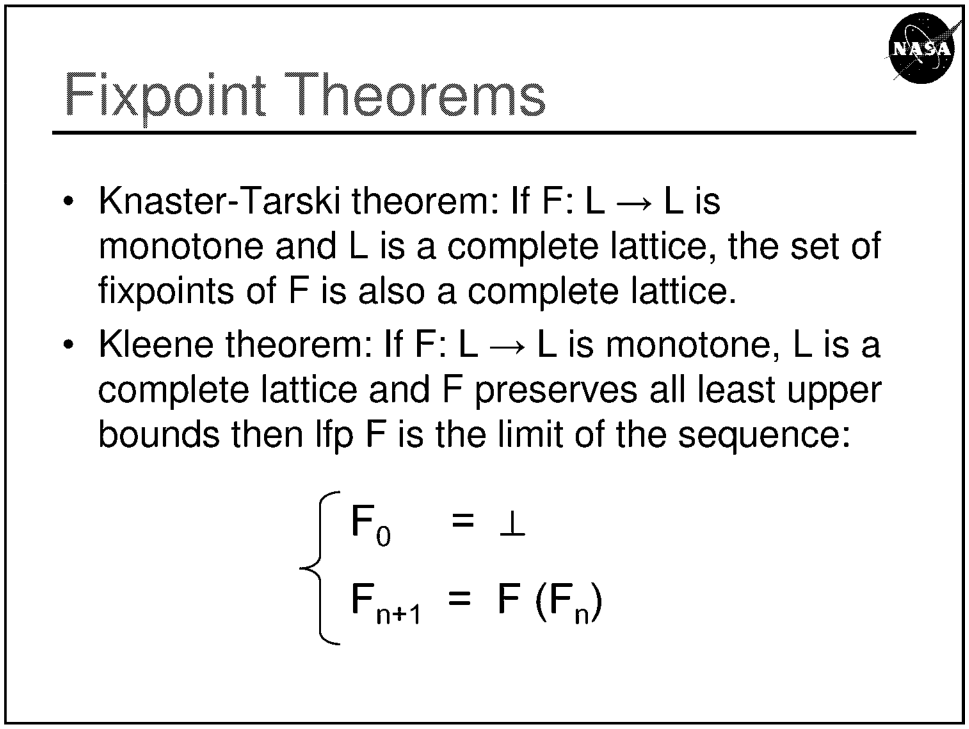
Methodology





Lattices and Fixpoints

- A lattice $(L, \sqsubseteq, \perp, \sqcup, \top, \sqcap)$ is a partially ordered set $(L, \sqsubseteq)$ with:
  - Least upper bounds ($\sqcup$) and greatest lower bounds ($\sqcup$) operators
  - A least element "bottom": $\perp$
  - A greatest element "top": $\top$
- L is complete if all least upper bounds exist
- A fixpoint X of $F: L \rightarrow L$ satisfies $F(X) = X$
- We denote by lfp F the least fixpoint if it exists

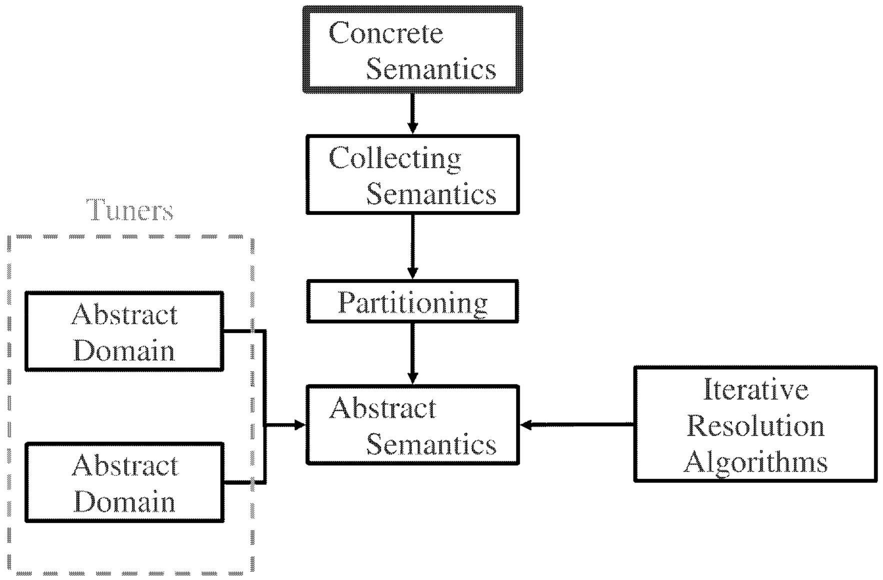


Fixpoint Theorems

- Knaster-Tarski theorem: If $F: L \rightarrow L$ is monotone and L is a complete lattice, the set of fixpoints of F is also a complete lattice.
- Kleene theorem: If $F: L \rightarrow L$ is monotone, L is a complete lattice and F preserves all least upper bounds then lfp F is the limit of the sequence:

$$\begin{cases} F_0 = \perp \\ F_{n+1} = F(F_n) \end{cases}$$

Methodology



Concrete Semantics

Small-step operational semantics: $(\Sigma, \rightarrow)$

$\boxed{s = \langle \boxed{\text{program point}}, \boxed{\text{env}} \rangle}$ $\boxed{s \rightarrow s'}$

<u>Example:</u>

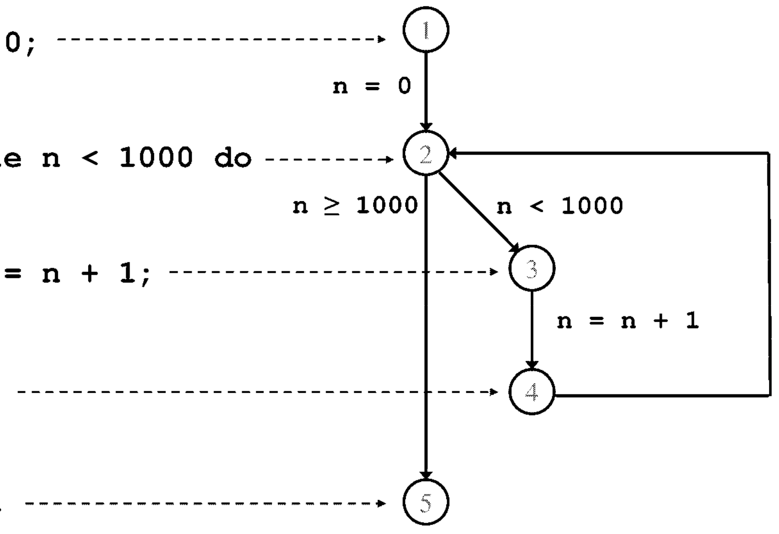
```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
```

$\langle 1, n \Rightarrow \Omega \rangle \rightarrow \langle 2, n \Rightarrow 0 \rangle \rightarrow \langle 3, n \Rightarrow 0 \rangle \rightarrow \langle 4, n \Rightarrow 1 \rangle$
$\rightarrow \langle 2, n \Rightarrow 1 \rangle \rightarrow \ldots \rightarrow \langle 5, n \Rightarrow 1000 \rangle$

Undefined value

Control Flow Graph

```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
```



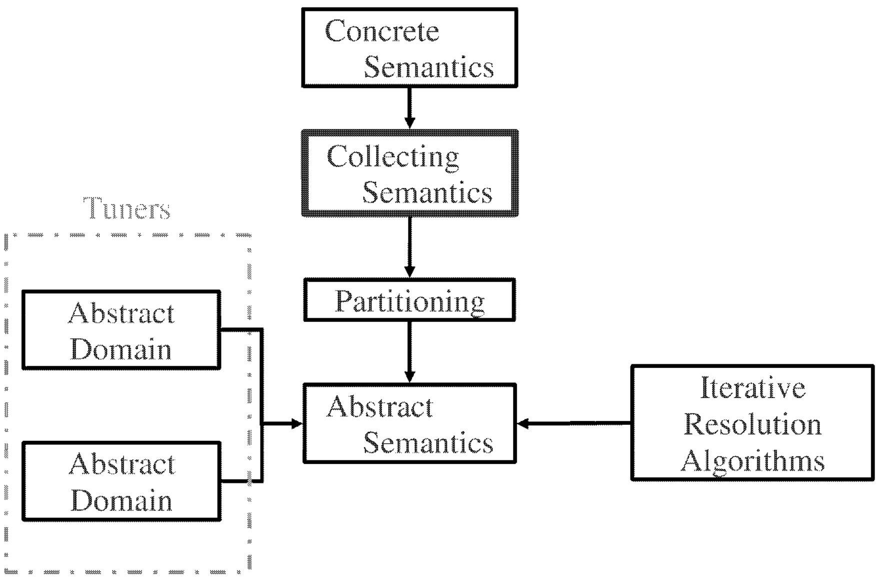
Transition Relation

Control flow graph: $\textcircled{i} \xrightarrow{\text{op}} \textcircled{j}$

Operational semantics: $\langle \textcircled{i}, \varepsilon \rangle \rightarrow \langle \textcircled{j}, [\![\text{op}]\!]\ \varepsilon \rangle$

Semantics of op

Methodology



Collecting Semantics

The collecting semantics is the set of observable behaviours in the operational semantics. It is the starting point of any analysis design.

- The set of all descendants of the initial state
- The set of all descendants of the initial state that can reach a final state
- The set of all finite traces from the initial state
- The set of all finite and infinite traces from the initial state
- etc.

Which Collecting Semantics?

- Buffer overrun, division by zero, arithmetic overflows: state properties
- Deadlocks, un-initialized variables: finite trace properties
- Loop termination: finite and infinite trace properties

State properties

The set of descendants of the initial state s_0:

$$\boxed{S = \{s \mid s_0 \to \ldots \to s\}}$$

<u>Theorem:</u> $F : (\wp(\Sigma), \subseteq) \to (\wp(\Sigma), \subseteq)$

$$F(S) = \{s_0\} \cup \{s' \mid \exists s \in S: s \to s'\}$$

$$\boxed{S = \mathrm{lfp}\ F}$$

Example

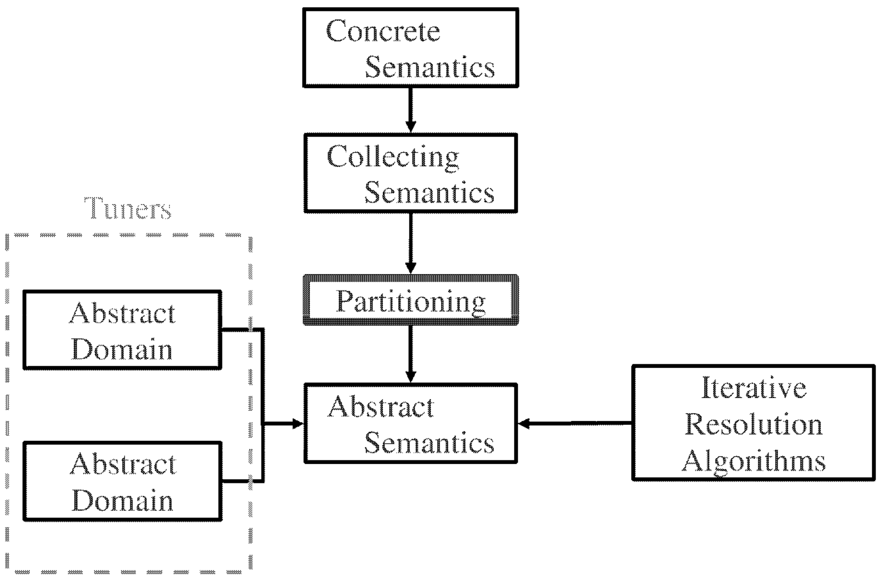
```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
```

$$S = \{\langle 1, n \Rightarrow \Omega\rangle, \langle 2, n \Rightarrow 0\rangle, \langle 3, n \Rightarrow 0\rangle, \langle 4, n \Rightarrow 1\rangle, \langle 2, n \Rightarrow 1\rangle, ..., \langle 5, n \Rightarrow 1000\rangle\}$$

Computation

- $F0 = \varnothing$
- $F1 = \{\langle 1,n\Rightarrow\Omega\rangle \}$
- $F2 = \{\langle 1,n\Rightarrow\Omega\rangle, \langle 2,n\Rightarrow 0\rangle \}$
- $F3 = \{\langle 1,n\Rightarrow\Omega\rangle, \langle 2,n\Rightarrow 0\rangle, \langle 3,n\Rightarrow 0\rangle \}$
- $F4 = \{\langle 1,n\Rightarrow\Omega\rangle, \langle 2,n\Rightarrow 0\rangle, \langle 3,n\Rightarrow 0\rangle, \langle 4,n\Rightarrow 1\rangle \}$
- ...

Methodology

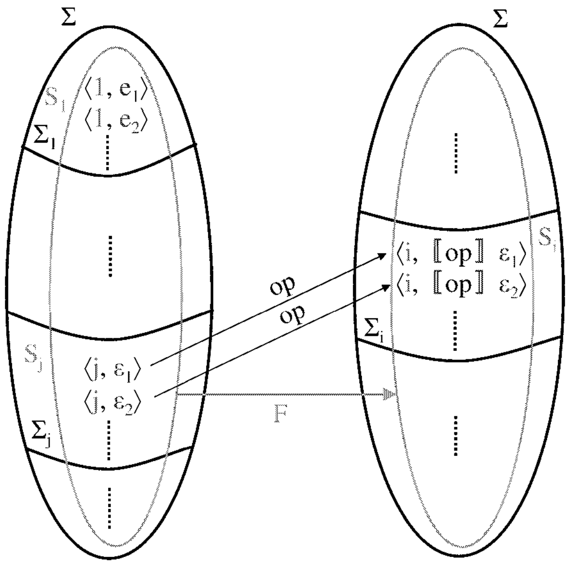


Partitioning

We partition the set S of states w.r.t. program points:

- $\Sigma = \Sigma_1 \oplus \Sigma_2 \oplus \ldots \oplus \Sigma_n$
- $\Sigma_i = \{\langle k, \varepsilon\rangle \in \Sigma \mid k = i\}$
- $F(S_1, \ldots, S_n)_i = \{s' \in S_i \mid \exists j\ \exists s \in S_j : s \to s'\}$
- $F(S_1, \ldots, S_n)_i = \{\langle i, [\![op]\!]\ \varepsilon\rangle \mid \text{ⓙ} \xrightarrow{op} \text{ⓘ} \in CFG(P)\}$
- $F(S_1, \ldots, S_n)_0 = \{ s_0 \}$

Illustration



Semantic Equations

- Notation: E_i = set of environments at program point i
- System of semantic equations:

$$\boxed{E_i = \bigcup \{ \llbracket op \rrbracket E_j \mid (j) \xrightarrow{op} (i) \in \mathrm{CFG}(P) \}}$$

- Solution of the system **= S =** lfp **F**

Example

```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
```

$E_1 = \{n \Rightarrow \Omega\}$

$E_2 = [\![\texttt{n = 0}]\!]\ E_1 \cup E_4$

$E_3 = E_2 \cap\ ]-\infty, 999]$

$E_4 = [\![\texttt{n = n + 1}]\!]\ E_3$

$E_5 = E_2 \cap [1000, +\infty[$

Example

```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
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$E_1 = \{n \Rightarrow \Omega\}$

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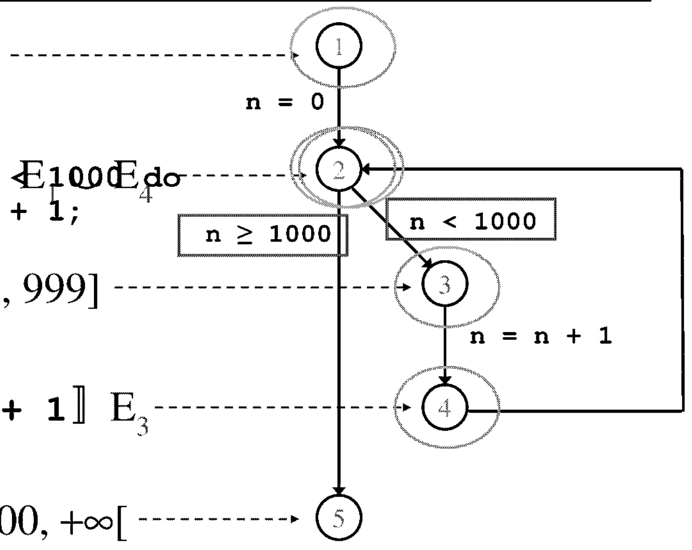
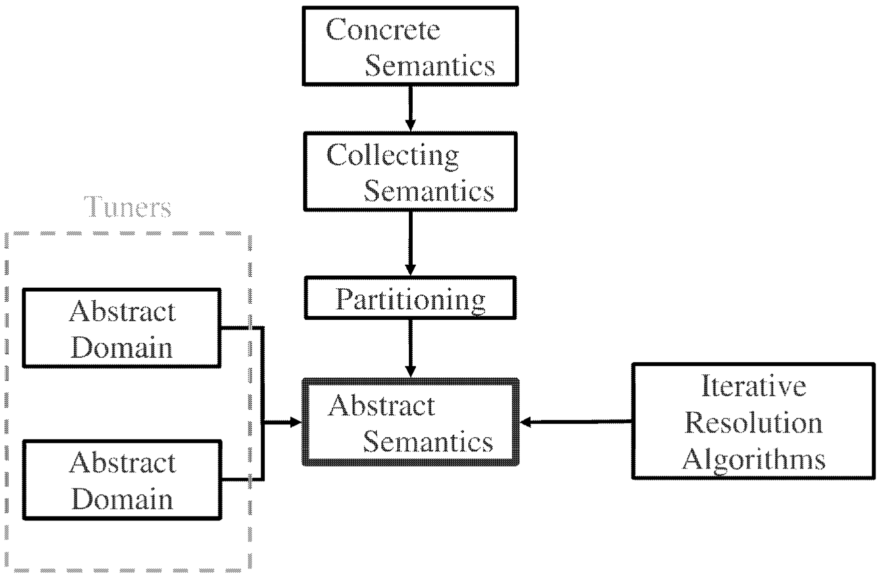


Other Kinds of Partitioning

- In the case of collecting semantics of traces:
  - Partitioning w.r.t. procedure calls: **context sensitivity**
  - Partitioning w.r.t. executions paths in a procedure: **path sensitivity**
  - Dynamic partitioning (Bourdoncle)

Methodology



Approximation

Problem: Compute a sound approximation $S^\#$ of S

$$\boxed{S \subseteq S^\#}$$

Solution: Galois connections

Galois Connection

L_1, L_2 two lattices

Abstract domain

$$\begin{tikzcd}(L_1, \subseteq) \arrow[r, shift right, "\alpha"'] & (L_2, \leq) \arrow[l, shift right, "\gamma"']\end{tikzcd}$$

- $\forall x \forall y : \alpha(x) \leq y \Leftrightarrow x \subseteq \gamma(y)$
- $\forall x \forall y : x \subseteq \gamma \circ \alpha(x) \ \& \ \alpha \circ \gamma(y) \leq y$

Fixpoint Approximation

$$\begin{tikzcd} L_2 \arrow[r, "\alpha \circ F \circ \gamma"] \arrow[d, "\gamma"'] & L_2 \\ L_1 \arrow[r, "F"'] & L_1 \arrow[u, "\alpha"'] \end{tikzcd}$$

<u>Theorem:</u>

$$\boxed{\text{lfp } F \subseteq \gamma(\text{lfp } \alpha \circ F \circ \gamma)}$$

Abstracting the Collecting Semantics

- Find a Galois connection:

$$(\wp(\Sigma), \subseteq) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} (\Sigma^{\#}, \leq)$$

- Find a function: $\alpha \circ F \circ \gamma \leq F^{\#}$

$$\boxed{\text{Partitioning} \Rightarrow \text{Abstract sets of environments}}$$

Abstract Algebra

- Notation: E the set of all environments
- Galois connection:

$$\begin{tikzcd}(\wp(E), \subseteq) \arrow[r, shift right, "\alpha"'] & (E^\#, \leq) \arrow[l, shift right, "\gamma"']\end{tikzcd}$$

- $\cup$, $\cap$ approximated by $\cup^\#$, $\cap^\#$
- Semantics $[\![\mathbf{op}]\!]$ approximated by $[\![\mathbf{op}]\!]^\#$

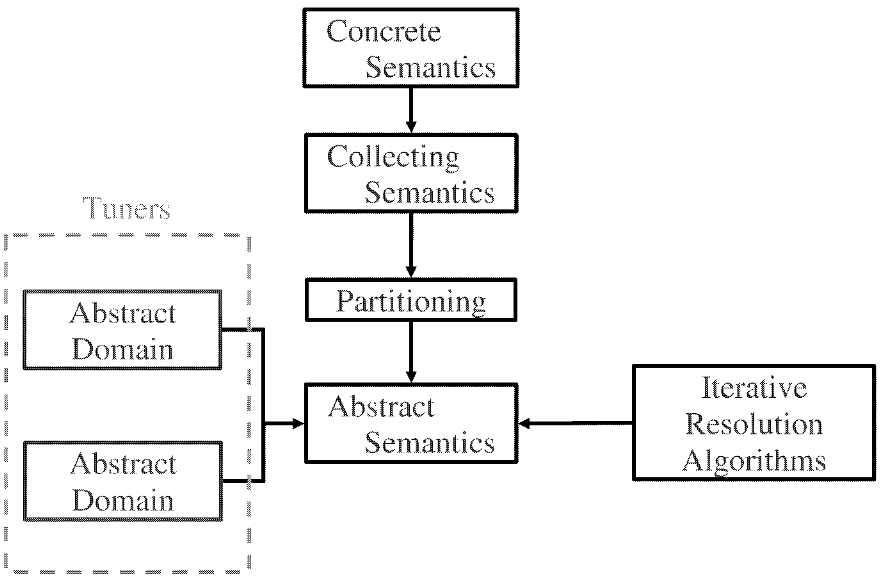
$$\boxed{\alpha \circ [\![\mathbf{op}]\!] \circ \gamma \subseteq [\![\mathbf{op}]\!]^\#}$$

Abstract Semantic Equations

```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
```

$$\boxed{\begin{array}{l}
E_1^\# = \alpha(\{n \Rightarrow \Omega\}) \\
E_2^\# = [\![\texttt{n = 0}]\!]^\# E_1^\# \cup^\# E_4^\# \\
E_3^\# = E_2^\# \cap^\# \alpha(]-\infty, 999]) \\
E_4^\# = [\![\texttt{n = n + 1}]\!]^\# E_3^\# \\
E_5^\# = E_2^\# \cap^\# \alpha([1000, +\infty[)
\end{array}}$$

Methodology



Abstract Domains

Environment: $x \Rightarrow v$, $y \Rightarrow w$, ...

Various kinds of approximations:

- Intervals (nonrelational):

  $x \Rightarrow [a, b]$, $y \Rightarrow [a', b']$, ...

- Polyhedra (relational):

  $x + y - 2z \leq 10$, ...

- Difference-bound matrices (weakly relational):

  $y - x \leq 5$, $z - y \leq 10$, ...

Example: intervals

```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit
```

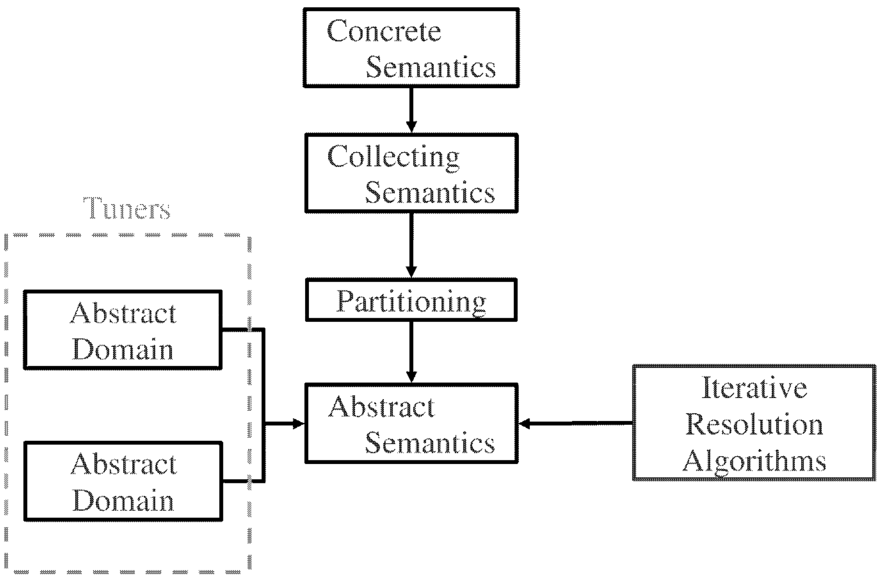
- Iteration 1: $E_2^\# = [0, 0]$
- Iteration 2: $E_2^\# = [0, 1]$
- Iteration 3: $E_2^\# = [0, 2]$
- Iteration 4: $E_2^\# = [0, 3]$
- ...

Problem

How to cope with lattices of infinite height?

Solution: automatic extrapolation operators

Methodology



Widening operator

Lattice (L, ≤): $\nabla : L \times L \to L$

- Abstract union operator:

$$\forall x \forall y : \; x \leq x \nabla y \;\; \& \;\; y \leq x \nabla y$$

- Enforces convergence: $(x_n)_{n \geq 0}$

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n \nabla x_{n+1} \end{cases}$$

$\boxed{(y_n)_{n\geq 0} \text{ is ultimately stationary}}$

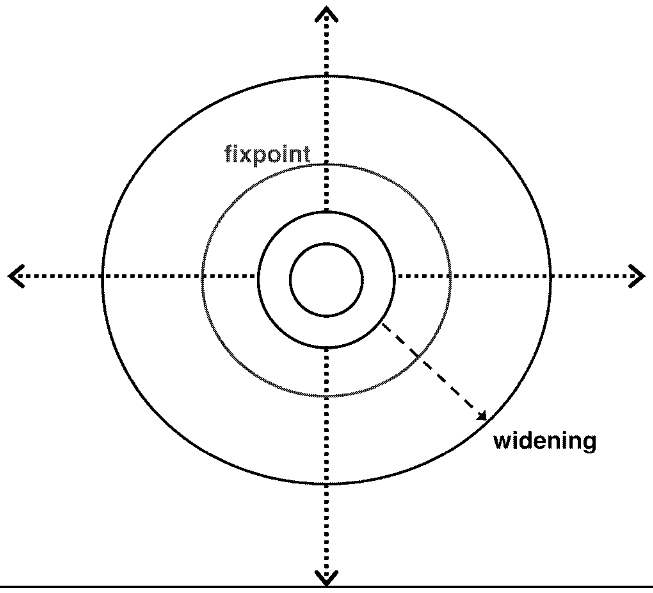
Widening of intervals

$$\boxed{[a, b] \nabla [a', b']}$$

- If $a \leq a'$ then a else $-\infty$
- If $b' \leq b$ then b else $+\infty$

↪ Open unstable bounds (jump over the fixpoint)

Widening and Fixpoint



Iteration with widening

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   exit
```

$$\boxed{(E_2^\#)_{n+1} = (E_2^\#)_n \nabla \left([\![\texttt{n = 0}]\!]^\# (E_1^\#)_n \cup^\# (E_4^\#)_n \right)}$$

Iteration 1 (union): $E_2^\# = [0, 0]$

Iteration 2 (union): $E_2^\# = [0, 1]$

Iteration 3 (widening): $E_2^\# = [0, +\infty] \Rightarrow$ stable

Imprecision at loop exit

```
1:   n = 0;
2:   while n < 1000 do
3:     n = n + 1;
4:   end
5:   ~~exit;~~ t[n] = 0; //t has 1500 elements
```

False positive!!!

- $E_5^\# = [1000, +\infty[$
- The information is present in the equations

Narrowing operator

Lattice $(L, \leq)$: $\Delta : L \times L \rightarrow L$

- Abstract intersection operator:

$$\forall x \forall y : x \cap y \leq x \,\Delta\, y$$

- Enforces convergence: $(x_n)_{n \geq 0}$

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n \,\Delta\, x_{n+1} \end{cases}$$

$\boxed{(y_n)_{n\geq 0} \text{ is ultimately stationary}}$

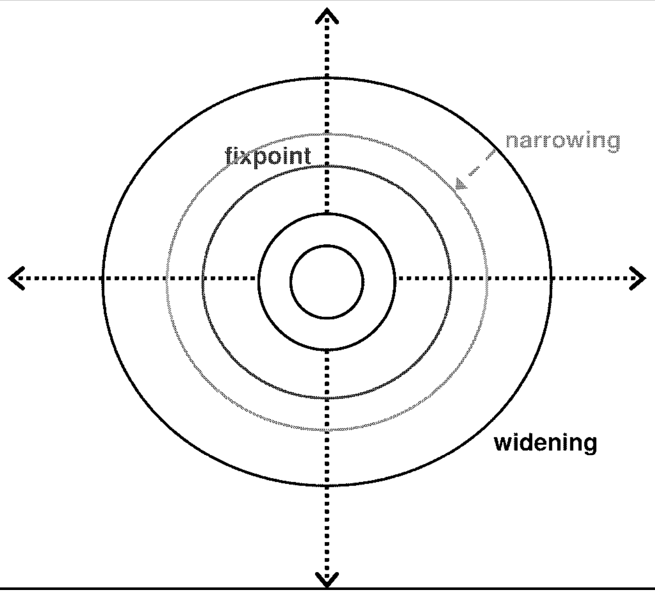
Narrowing of intervals

$\boxed{[a, b] \,\Delta\, [a', b']}$

- If $a = -\infty$ then a' else a
- If $b = +\infty$ then b' else b

↳ Refine open bounds

Narrowing and Fixpoint



Iteration with narrowing

```
1:  n = 0;
2:  while n < 1000 do
3:    n = n + 1;
4:  end
5:  exit; t[n] = 0;
```

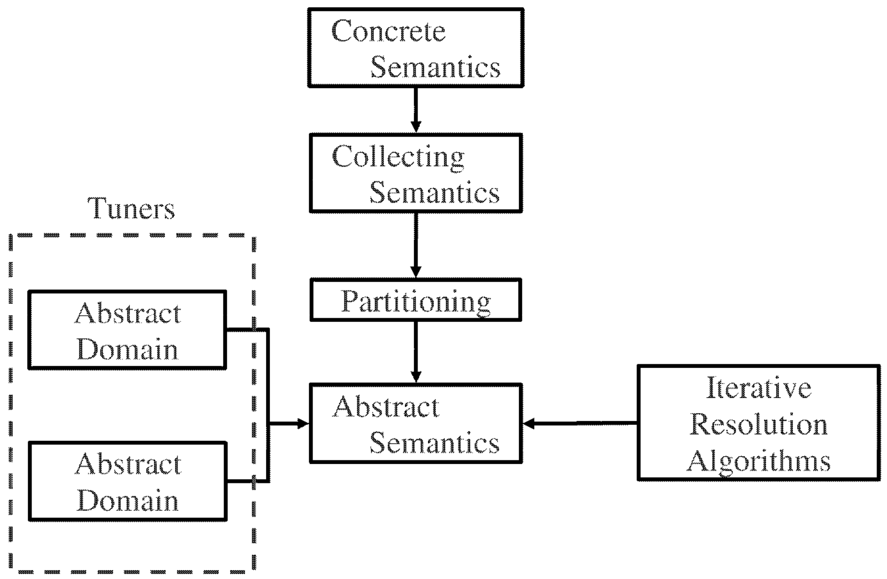
$$(E_2^{\#})_{n+1} = (E_2^{\#})_n \,\Delta \left([\![\texttt{n = 0}]\!]^{\#} (E_1^{\#})_n \cup^{\#} (E_4^{\#})_n \right)$$

Beginning of iteration: $E_2^{\#} = [0, +\infty[$

Iteration 1: $E_2^{\#} = [0, 1000] \Rightarrow$ stable

Consequence: $E_5^{\#} = [1000, 1000]$

Methodology



Tuning the abstract domains

```
n = 0;
k = 0;
while n < 1000 do
  n = n + 1;
  k = k + 1;
end
exit
```

- Intervals:

$$E_4^{\#} = \langle\, n \Rightarrow [0, 1000],\ k \Rightarrow [0, +\infty[\, \rangle$$

- Convex polyhedra or DBMs:

$$E_4^{\#} = \langle\, 0 \le n \le 1000,\ 0 \le k \le 1000,\ n - k = 0 \,\rangle$$

Comparison with Data Flow Analysis

Data Flow Framework

- Forward Data Flow Equations

$$\mathrm{in}(B) = \begin{cases} Init & ,\mathrm{B} = \mathrm{entry} \\ \bigcap\limits_{P \in \mathrm{Pred}(B)} \mathrm{F}_B(\mathrm{in}(B)) & ,\mathrm{otherwise} \end{cases}$$

- L is a lattice
- in(*B*) ∈ L is the data-flow information on entry to *B*
- *Init* is the appropriate initial value on entry to the program
- F_B is the transformation of the data-flow information upon executing block *B*
- ∩ models the effect of combining the data-flow information on the edges entering a block

Data-Flow Solutions

- Solving the data-flow equations computes the meet-over-all-paths (MOP) solution

$$\boxed{\mathrm{MOP}(B) = \bigcap_{p \in \mathrm{Path}(B)} F_p(\mathit{Init}) \text{ for } B = \mathit{entry}, B1, \ldots, Bn, \mathit{exit}}$$

- If F_B is monotone, i.e., $\boxed{F_B(x \cap y) \subseteq F_B(x) \cap F_B(y)}$
- then MOP ≤ MFP (maximum fixpoint)
- If F_B is distributive, i.e., $\boxed{F_B(x \cap y) = F_B(x) \cap F_B(y)}$
- then MOP = MFP

Typical Data-Flow Analyses

- Reaching Definitions
- Available Expressions
- Live Variables
- Upwards-Exposed Uses
- Copy-Propagation Analysis
- Constant-Propagation Analysis
- Partial-redundancy Analysis

Reaching Definitions

- Data-flow equations:

  $\forall i$: RCHin(i) = U (GEN(j) $\cup$ (RCHin(j) $\cap$ PRSV(j)))

where
  - PRSV are the definitions preserved by the block
  - GEN are the definitions generated by the blocks

- This is an iterative forward bit-vector problem
  - Iterative: it is solved by iteration from a set of initial values
  - Forward: information flows in direction of execution
  - Bit-vector: each definition is represented as a 1 (may reach given point) or a 0 (it does not reach this point)

AI versus classical DFA

- Classical DFA is stated in terms of properties whereas AI is usually stated in terms of models, whence the duality in the formulation.
- In classical DFA the proof of soundness must be made separately whereas it comes from the construction of the analysis in AI.
- Added benefits of AI:
  - Approximation of fixpoints in AI
    - Widening operators
    - Narrowing operators
  - Abstraction is explicit in AI
    - Galois connections
    - Can build a complex analysis as combination of basic, already-proved-correct, analyses

Annotated Bibliography

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